> 80 has the equation

$$\epsilon = 2.66\ [(x/s)\lambda^{-0.8}]^{-0.22}$$

Thus, the cooling length ($\epsilon = 1$) is expressed as

$$x_{CL}/s = 85\lambda^{0.8}$$

where the cooling length (x_{CL}/s) is the maximum value of x/s for which the surface temperature downstream of the slot is equal to or less than the total temperature of the slot flow ($T_{t,j}$). Included in the figure is the prediction from an implicit finite-difference method[4]; this method was applied by Bushnell and Beckwith to slot flow in a manner similar to that in Ref. 5 except that the mixing-length distribution was modified to account for the slot flow and mixing-region downstream of the slot. Predictions from Ref. 5 apply only far downstream of the slot, whereas the present prediction applies close to the slot location as well. The present prediction is for $s = 0.159$ cm and the static pressure at the slot exit equal to the freestream static pressure ("matched" pressure condition). Experimental data for the matched pressure condition correlate well with data for higher slot exit pressure to freestream static pressure ratios when compared as in Fig. 2, and the finite-difference prediction agrees with the level and trend of the present data.

The only other investigations of hypersonic film cooling effectiveness with tangential injection were also at Mach 6. A fairing of the film cooling effectiveness data obtained from heat-transfer measurements in Ref. 2 generally falls below the present results as shown in Fig. 2. The disagreement between the two sets of data is unexpected since the flow conditions were similar. Part of the difference in the correlations may be caused by the data reduction technique used in Ref. 2 (equilibrium temperatures were inferred from surface heat-transfer measurements) and part to the difference in experimental configurations (a slight injection angle for Ref. 2). A fairing of effectiveness obtained from unpublished direct measurements of surface equilibrium temperature (supplied by V. Zakkay of New York University) for the same conditions and configuration as Ref. 2 is also shown in Fig. 2. Although the effectiveness obtained from direct temperature measurements using the configuration of Ref. 2 better agrees with the present results than the fairing from Ref. 2, a significant disagreement still exists for $(x/s)\lambda^{-0.8} > 200$. Included in Fig. 2 for comparison is a fairing of data from Ref. 3 for tangential supersonic slot injection (slot Mach number at the slot exit equal to 2.3) into a Mach 6 stream. The slope of the present data substantially agrees with that from Ref. 3 for $(x/s)\lambda^{-0.8} > 300$. Although significant differences exist between the values of film cooling effectiveness found herein and from other hypersonic investigations, all the results show more efficiency for film cooling in hypersonic turbulent flow than would be inferred by extrapolation of subsonic results.

In summary, this investigation of the effect of sonic slot injection into a thick turbulent-boundary layer on the downstream equilibrium temperature confirms that film cooling in hypersonic flow is significantly more efficient than previous extrapolations of lower-speed results had indicated. In addition, the present results indicate a greater film cooling effectiveness than found in other hypersonic investigations. Thus, a new evaluation of film cooling as an active cooling system for use on hypersonic vehicles is desirable.

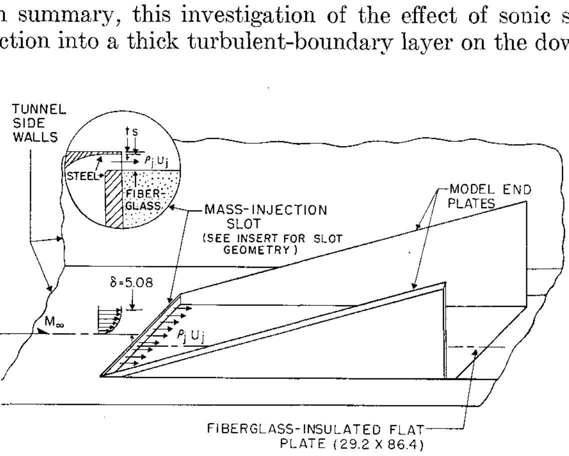

Fig. 1 Two-dimensional film cooling model; all dimensions in centimeters.

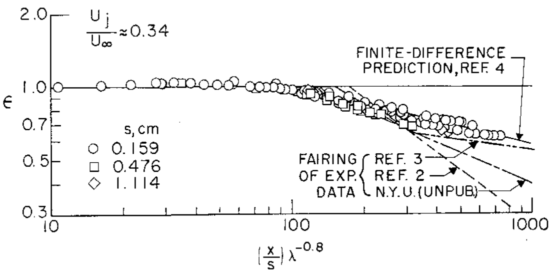

Fig. 2 Film cooling effectiveness at Mach 6.

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An Improved Conjugate Direction Minimization Procedure

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THE more advanced parameter optimization processes, e.g., Davidon-Fletcher-Powell,[1] are based upon generation of conjugate direction sequences and/or inference of the second partial derivative matrix inverse. A method having both of these properties and, in addition, being free of one-dimensional minimization requirements would be advantageous. In attempting to synthesize such a method, the writers have come to recognize that a "batch processor" version of DFP considered in an earlier study[2] does, in fact, combine the desirable properties by mere deletion of the one-dimensional search. This version will be reviewed, and an argument advanced for conjugacy of the directions generated in the ab-

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Table 1 Quadratic test results with golden section search

Search tolerance ϵ	DFP f	DFP Number of gradient evaluations	DFP Number of function evaluations	Modification f	Modification Number of gradient evaluations	Modification Number of function evaluations
10^{-4}	10^{-37}	9	228	10^{-35}	8	205
10^{-3}	10^{-29}	9	184	10^{-34}	8	166
10^{-2}	10^{-39}	10	154	10^{-31}	8	128
10^{-1}	10^{-25}	11	111	10^{-29}	8	91
$10^{0} \leq \epsilon$	10^{-20}	14	89	10^{-25}	8	61

sence of 1-D searches. For a quadratic function of n-parameters, inference of the second partial derivative matrix in n steps implies $n + 1$-step convergence. Results of a limited computational comparison will be presented.

Minimization Algorithm

The function $f(x)$ is to be minimized by computation of the gradient f_x, and generation of steps Δx by the formula

$$\Delta x = -\alpha H f_x \tag{1}$$

Here x is an n-vector and H an $n \times n$ matrix, initially positive definite and symmetric, updated by

$$H + \Delta H = H - (H\Delta f_x \Delta f_x{}^T H)/(\Delta f_x{}^T H \Delta f_x) \tag{2}$$

after each step. The increment in the gradient vector is Δf_x. H is reduced in rank, becoming null after the nth update, when it is replaced by

$$\Sigma \Delta x \Delta x^T/(\Delta f_x{}^T \Delta x) \tag{3}$$

The summation extends over n terms, and represents an estimate of $f_{xx}{}^{-1}$, exact for quadratic f and conjugate steps.

Conjugacy and Convergence

Conjugacy of the direction sequence for arbitrary choice of the step-size parameters α proceeds according to the following argument. By definition of conjugacy:

$$\Delta x_i{}^T f_{xx} \Delta x_j = 0 \qquad i \neq j \tag{4}$$

For nonquadratic functions, a tentative, if somewhat unsatisfactory, definition is given by

$$\Delta f_{x_i}{}^T \Delta x_j = 0 \qquad j > i \tag{5}$$

With H_o the initial metric choice,

$$H_1 = H_o - (H_o \Delta f_{x1} \Delta f_{x1}{}^T H_o)/(\Delta f_{x1}{}^T H_o \Delta f_{x1}) \tag{6}$$

$$H_{k+1} = H_k - (H_k \Delta f_{xk+1} \Delta f_{xk+1}{}^T H_k)/\Delta f_{xk+1}{}^T H_k \Delta f_{xk+1} \tag{7}$$

By direct evaluation, one obtains

$$\Delta f_{x1}{}^T H_1 = 0 \tag{8}$$

$$\Delta f_{x1}{}^T H_2 = 0 \tag{9}$$

$$\Delta f_{x2}{}^T H_2 = 0 \tag{10}$$

Table 2 Nonquadratic test results with golden section search

Search tolerance ϵ	DFP f Significant figures	DFP Number of gradient evaluations	DFP Number of function evaluations	Modification f Significant figures	Modification Number of gradient evaluations	Modification Number of function evaluations
10^{-4}	16	29	746	13	38	914
10^{-3}	16	33	707	13	39	751
10^{-2}	16	53	899	11	38	536
10^{-1}	16	201	2392	12	42	390
10^{0}	16	78	539	7	41	261

and it follows by induction that

$$\Delta f_{x_i}{}^T H_j = 0 \qquad j \geq i \tag{11}$$

so that conjugacy

$$\Delta f_{x_i}{}^T \Delta x_j = -\Delta f_{x_i}{}^T H_{j-1} f_{x_i}{}^T H_{j-i} f_{x_{j-i}} \alpha_j = 0 \qquad j_{x_{ji}} > i \tag{12}$$

results independently of the α choices.

The n-step convergence of the DFP method is lost because conjugacy of the directions, by itself, is not enough without one-dimensional minimizations; however, the exact estimate of $f_{xx}{}^{-1}$ given by Eq. (3) produces the minimum for $\alpha = 1$ on the $n + 1$st step, for quadratic f.

Numerical Experiments

Some numerical results will be presented comparing DFP with the proposed modification as the 1-D search is coarsened. The computations use two examples from Ref. 2: the first, a quadratic of no particular difficulty and the second, a propellant minimization problem for orbit transfer using a conic/impulse model with the formulation of Ref. 3. The search routine employed is the "golden section" scheme of Ref. 4, an efficient bracketing technique.

Results for the quadratic function of dimension $n = 7$ are shown in Table 1. The search tolerance figure ϵ represents the allowable difference between neighboring function samples at search termination.

Corresponding results for the nonquadratic test problem, also of dimension 7, are shown in Table 2. The quadratic and nonquadratic examples have the same second partial derivatives at the minimum.

With a high-accuracy search, DFP out-performs the modified version as a result of having "fresher" second partial derivative information in use. Deterioration in DFP proceeds nearly to the point of loss of H-definiteness, for $\epsilon = 10^{-1}$. This is fortuitous, since definiteness can be easily lost in DFP without accurate 1-D minimizations and complete convergence failure experienced. The modified algorithm is immune to this and convergence is little affected as the search tolerance is loosened. The loss in accuracy of defining the minimum with the modified process is thought to be due to the enlarged null space of H as given by Eq. (7) toward the end of a batch, which tends to trigger the termination criterion prematurely. Possibly, the most recent full-rank H matrix should be retained after a certain point has been reached, e.g., when the gradient vector has become so small that the estimate $\frac{1}{2}\Delta f_x{}^T H \Delta f_x$ of the possible further reduction in f affects only the qth significant figure. Exploration of this point is of future interest, as is the evaluation of other simplified search schemes.

Discussion and Conclusion

The number of algorithms designed to have quadratic terminal convergence has grown so that even tabulation would be a considerable task, and comparison a major undertaking. The general impression, however, is that those depending purely on conjugacy have the disadvantage of open-loop operation with general nonquadratic functions, whereas those concentrating on $f_{xx}{}^{-1}$ inference are subject to numerical error magnification unless independence of steps is assured, as via conjugacy. DFP is thus hard to beat because of the combination of conjugacy and $f_{xx}{}^{-1}$ inference. The motivation for the presently proposed process, which retains this combination but economizes on searching, was prompted by experience in various flight mechanics applications which underscored the costliness of one-dimensional searching.

The limited computational results show what one might expect a priori from the properties of the competing methods. Certainly a comprehensive comparison of many contenders in a variety of examples would be worthwhile and, until someone has undertaken this, any conclusions must be tentative. The general high performance of DFP in various past comparisons, however, tends to recommend any scheme really competitive with it.

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A Discrete Search Procedure for the Minimization of Stiffened Cylindrical Shell Stability Equations

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Introduction

THE analytical methods conventionally used for treating the stability of stiffened cylindrical shells under uniform axial compression and/or lateral pressure require the minimization of the buckling equations with respect to the number of axial half-waves (m) and circumferential full waves (n) where n and m are integers.[1-3] The usual procedures seem to be an exhaustive discrete search or a method combining an iterative minimization with respect to n with a discrete exhaustive search of m.[1,3] Timoshenko and Gere[4] discuss minimization methods for unstiffened and ring stiffened shells. Such procedures, although adequate where only a few designs are analyzed, would be inefficient if applied to shell synthesis, where many designs must be evaluated.[1,5-8]

This Note presents a more efficient, discrete, search method for treating this problem and includes a discussion of the nature of the buckling load functions for three lateral pressure loading conditions.

Minimization Problem

Consider the problem of finding the distributed unit axial compressive buckling load for a stiffened cylinder under uniform lateral pressure. For a given set of design parameters and pressure the buckling function $N_{cl}(n,m)$ has two discrete, independent variables, the axial and circumferential wave numbers m and n [see for example Eq. (31) of Ref. 2 and Eq. (29) of Ref. 3]. To obtain the buckling load, therefore, one must find the minimum of $N_{cl}(n,m)$ with respect to $n = 0,1,2,\ldots$, and $m = 1,2,3,\ldots$. This formulation can be viewed as an unconstrained discrete optimization problem and treated by an appropriate mathematical programming procedure.

The N_{cl} function can be represented by a surface in three-dimensional space if, for the purposes of illustration only, n and m are considered continuous. Contour maps of typical buckling load surfaces for the three possible pressure loading situations are shown in Figs. 1–3, where $N(n,m) = N_{cl}/N_{cl}^*$ and N_{cl}^* is the discrete minimum of N_{cl}. These are buckling values for a simply supported ring-stringer stiffened shell calculated using equations given by Burns.[1] It may be seen that in the absence of lateral pressure the surface is unimodal and a region of strong interaction exists between n and m with a resulting resolution ridge, starting at the $m = 1$ boundary. This ridge is encountered in most designs and can pose a serious problem for many search procedures. All surfaces of this type are, fortunately, unimodal.

Where there is significant internal pressure (Fig. 2), the unimodality remains but the resolution ridge typically vanishes. For the case of significant external pressure the surface is typically bimodal.[1] One of the local minima, the

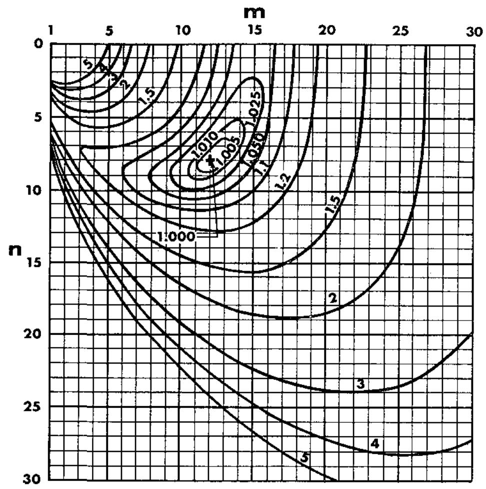

Fig. 1 Typical contour map of the $N(n, m)$ surface for a shell under no lateral pressure.

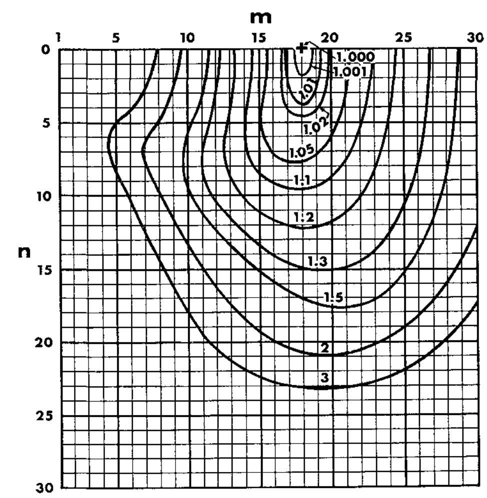

Fig. 2 Typical contour map of the $N(n, m)$ surface for a shell under significant internal pressure.

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